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by

Patrick J. Blewett

METHOD OF COMPUTING FLUID MOTION IN TWO-DIMENSIONAL CARTESIAN OR CYLINDRICAL COORDINATES BY FOLLOWING LAGRANGIAN ENERGY CELLS

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ABSTRACT

It is conjectured that for some physical flow problems there may be an advantage in following energy exactly as one follows mass elements in the conventional Lagrangian formulation of fluid dynamics. Essentially, the argument is a generalization to two dimensions of an idea due to Enig who derived the equations for one-dimensional slab geometry. The equations are derived both differentially and integrally from the Eulerian form, and they are compared with the conventional Lagrangian equations.

NOTATION

- e specific internal energy
- E total energy in Lagrangian energy cell, $\rho_o\psi_o V_o$
- J Jacobian of transformation, $r_\xi z_\eta - r_\eta z_\xi$
- J' $(r/\xi)^\alpha J$
- P pressure
- r radial coordinate
- t time
- u radial speed
- V volume
- $\vec{v}$ velocity vector, $(u,0,w)$
- w axial speed
- z axial coordinate
- α 1 or 0, for cylindrical or Cartesian plane geometries, respectively
- η Lagrangian mass or energy coordinate, $z(\xi,\eta,0)$
- ξ Lagrangian mass or energy coordinate, $r(\xi,\eta,0)$
- ρ density
- ψ total specific energy, $\frac{1}{2}(u^2+w^2)+e$
- o subscript denoting initial values

A letter subscript denotes partial differentiation.

This report derives and examines the equations governing the motion of energy elements in cylindrical coordinates where symmetry in Θ is assumed. This is analogous to the conventional Lagrangian formulation of mass motion from Euler's fundamental equations. For physical problems involving no energy sources, such as impact problems, or for certain problems wherein a time is reached at which there is no further release of energy, it is conjectured that there may be some advantage in following the flow of energy in addition to mass flow. Computations of impact problems in two-dimensional fluid mechanics in the conventional Lagrangian formulation often encounter difficulties of mesh distortion, partly because of the impenetrability of mass. Because energy, however, can flow across a collision surface, it might be possible to compute its motion more smoothly by following such elements. Enig[1] derived the equations for one-dimensional slab geometry and computed from them the problem of an infinite gas moving toward an infinitely rigid wall. He achieved good agreement with the analytic solution. To my knowledge, Enig's is the only application of this method.

We will derive the equations differentially, leaving most of the mathematical details to the appendixes, and will interpret the resulting equations both differentially and integrally.

We start with Euler's formulation of the equations of conservation of mass, energy, and momentum, and attempt to keep cylindrical coordinates separate from two-dimensional Cartesian coordinates

$$\rho_t + \nabla\cdot\rho\vec{v} = 0, \qquad \text{mass}, \tag{1}$$

$$(\rho\psi)_t + \nabla\cdot[(\rho\psi+P)\vec{v}] = 0, \qquad \text{energy}, \tag{2}$$

$$(\rho u)_t + \nabla\cdot(\rho u\vec{v}) + P_r = 0, \qquad r \text{ momentum}, \tag{3}$$

$$(\rho w)_t + \nabla\cdot(\rho w\vec{v}) + P_z = 0, \qquad z \text{ momentum}, \tag{4}$$

or:

$$\rho_t + \frac{\alpha}{r}\rho u + (\rho u)_r + (\rho w)_z = 0, \tag{5}$$

$$(\rho\psi)_t + \frac{\alpha}{r}(\rho\psi+P)u + [(\rho\psi+P)u]_r + [(\rho\psi+P)w]_z = 0, \tag{6}$$

$$(\rho u)_t + \frac{\alpha}{r}\rho u^2 + (\rho u^2+P)_r + (\rho uw)_z = 0, \tag{7}$$

$$(\rho w)_t + \frac{\alpha}{r}\rho wu + (\rho wu)_r + (\rho w^2+P)_z = 0. \tag{8}$$

In the conventional way, we now let

$$r \to r(\xi,\eta,t), \tag{9}$$

$$z \to z(\xi,\eta,t). \tag{10}$$

Under this transformation, the partial-derivative operators transform as

$$\frac{\partial}{\partial r} \to \frac{1}{J}\left(z_\eta \frac{\partial}{\partial\xi} - z_\xi \frac{\partial}{\partial\eta}\right), \tag{11}$$

$$\frac{\partial}{\partial z} \to \frac{1}{J}\left(r_\xi \frac{\partial}{\partial\eta} - r_\eta \frac{\partial}{\partial\xi}\right), \tag{12}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{1}{J}\left[(r_t z_\xi - r_\xi z_t)\frac{\partial}{\partial\eta} + (r_\eta z_t - r_t z_\eta)\frac{\partial}{\partial\xi}\right], \tag{13}$$

where

$$J = r_\xi z_\eta - r_\eta z_\xi. \tag{14}$$

Equations (11) through (13) are derived in Appendix A.

Applying Eqs. (11) through (13) on Eq. (5), we have

$$\rho_t J + (r_t z_\xi - r_\xi z_t)\rho_\eta + (r_\eta z_t - r_t z_\eta)\rho_\xi + \alpha J\frac{u}{r}$$
$$+ z_\eta(\rho u)_\xi - z_\xi(\rho u)_\eta + r_\xi(\rho w)_\eta - r_\eta(\rho w)_\xi = 0.$$

Into the above equation we substitute

$$\rho_t J = (\rho J)_t - \rho(r_\xi z_{\eta t} + r_{\xi t} z_\eta - r_\eta z_{\xi t} - r_{\eta t} z_\xi),$$

and collect terms in r_ξ, r_η, z_η, z_ξ. The result is

$$(\rho J)_t + \alpha J\frac{\rho u}{r} + r_\xi\left[-\rho z_{\eta t} - \rho_\eta z_t + (\rho w)_\eta\right]$$
$$+ r_\eta\left[\rho z_{\xi t} + \rho_\xi z_t - (\rho w)_\xi\right] + z_\eta\left[-\rho r_{\xi t} - \rho_\xi r_t + (\rho u)_\xi\right]$$
$$+ z_\xi\left[\rho r_{\eta t} + \rho_\eta r_t - (\rho u)_\eta\right] = 0.$$

This equation can then be put in the more meaningful form,

$$(\rho J)_t + \alpha J\frac{\rho u}{r} = r_\xi[\rho(z_t - w)]_\eta - r_\eta[\rho(z_t-w)]_\xi$$
$$+ z_\eta[\rho(r_t-u)]_\xi - z_\xi[\rho(r_t-u)]_\eta. \tag{15}$$

We note here that Eq. (15) is identically satisfied by

$$\rho J = \left(\frac{\xi}{r}\right)^\alpha \rho_0(\xi,\eta), \tag{16}$$

$$r_t = u, \tag{17}$$

$$z_t = w, \tag{18}$$

because the left-hand side of Eq. (15) with the aid of Eq. (16) is

$$(\rho J)_t + \alpha J\frac{\rho u}{r} = \rho_0\,\alpha\left(\frac{\xi}{r}\right)^{\alpha-1}\left(-\frac{\xi}{r^2}\right)r_t + \alpha\rho_0\left(\frac{\xi}{r}\right)^\alpha\frac{u}{r},$$

or

$$(\rho J)_t + \alpha J\frac{\rho u}{r} = -\rho_0\,\alpha\left(\frac{\xi}{r}\right)^\alpha\frac{1}{r}(r_t - u).$$

Now looking back at Eqs. (6), (7), and (8), we see that their forms are similar to Eq. (5), namely, a term differentiated with respect to t, a term in α due to radial motion in cylindrical coordinates, and then two terms differentiated, respectively, with r and z. We leave to Appendix B the repetitive algebra of applying Eqs. (11) through (13) to Eqs. (6) through (8) to derive equations like Eq. (15). With Eqs. (15) the results are

$$(\rho J)_t + \alpha J\frac{\rho u}{r} = r_\xi[\rho(z_t-w)]_\eta - r_\eta[\rho(z_t-w)]_\xi$$
$$+ z_\eta[\rho(r_t-u)]_\xi - z_\xi[\rho(r_t-u)]_\eta \qquad \text{mass}, \tag{19}$$

$$(\rho J\psi)_t + \alpha J(\rho\psi+P)\frac{u}{r} = r_\xi[\rho\psi(z_t-w)-Pw]_\eta - r_\eta[\rho\psi(z_t-w)-Pw]_\xi$$
$$+ z_\eta[\rho\psi(r_t-u)-Pu]_\xi - z_\xi[\rho\psi(r_t-u)-Pu]_\eta \qquad \text{energy}, \tag{20}$$

$$(\rho J u)_t + \alpha J \frac{\rho u^2}{r} = r_\xi[\rho u(z_t - w)]_\eta - r_\eta[\rho u(z_t - w)]_\xi$$

$$+ z_\eta[\rho u(r_t - u) - P]_\xi - z_\xi[\rho u(r_t - u) - P]_\eta \quad \text{r momentum,} \quad (21)$$

and

$$(\rho J w)_t + \alpha J \frac{\rho w u}{r} = r_\xi[\rho w(z_t - w) - P]_\eta - r_\eta[\rho w(z_t - w) - P]_\xi$$

$$- z_\eta[\rho w(r_t - u)]_\xi - z_\xi[\rho w(r_t - u)]_\eta \quad \text{z momentum.} \quad (22)$$

Let us consider what has been done. We started with the four Eulerian equations, (5) through (8), in the five dependent variables, ρ, u, w, P, and e, and the three independent variables, r, z, and t. We assume that we also have an equation of state relating ρ, P, and e so that we have five equations in five unknowns. With the transformations, Eqs. (9) and (10), we have created two more dependent variables, r and z; hence, Eqs. (19) through (22), complemented with an equation of state, constitute five equations in seven unknowns in the new independent variables ξ, η, and t. In the conventional Lagrangian formulation, Eq. (19) is identically satisfied by replacing it with Eqs. (16), (17), and (18). Thus, we have seven equations in the seven unknowns, r, z, u, w, ρ, P, and e. With the aid of Eqs. (16) and (17), we note that

$$(\rho J \phi)_t + \alpha J \rho \phi \frac{u}{r} = \left(\frac{\xi}{r}\right)^\alpha \rho_o \phi_t - \alpha\left(\frac{\xi}{r}\right)^\alpha \rho_o \frac{1}{r}(r_t - u)\phi,$$

or

$$(\rho J \phi)_t + \alpha J \rho \phi \frac{u}{r} = \left(\frac{\xi}{r}\right)^\alpha \rho_o \phi_t \ , \quad (23)$$

where ϕ is any dependent variable. Thus with Eqs. (23), (17), and (18), Eqs. (20), (21), and (22) simplify to:

$$\rho_o\left[\psi_t + \alpha \frac{Pu}{\rho r}\right] = \left(\frac{r}{\xi}\right)^\alpha[-r_\xi(Pw)_\eta + r_\eta(Pw)_\xi - z_\eta(Pu)_\xi + z_\xi(Pu)_\eta]$$

$$\text{energy,} \quad (24)$$

$$\rho_o u_t = \left(\frac{r}{\xi}\right)^\alpha[-z_\eta P_\xi + z_\xi P_\eta] \quad \text{r momentum,} \quad (25)$$

and

$$\rho_o w_t = \left(\frac{r}{\xi}\right)^\alpha[-r_\xi P_\eta + r_\eta P_\xi] \quad \text{z momentum.} \quad (26)$$

The above three equations are recognized as the conventional Lagrangian formulas in two dimensions.

To follow energy elements, instead of mass elements, let us ask what are the consequences of satisfying Eq. (20) identically. To accomplish this, let

$$\rho J \psi = \rho_o(\xi,\eta)\psi_o(\xi,\eta)\left(\frac{\xi}{r}\right)^\alpha . \quad (27)$$

Then,

$$(\rho J \psi)_t = -\rho_o \psi_o \alpha\left(\frac{\xi}{r}\right)^\alpha \frac{r_t}{r} , \quad (28)$$

and the left-hand side of Eq. (20) becomes

$$-\rho_o \psi_o \alpha\left(\frac{\xi}{r}\right)^\alpha \frac{1}{r}\left[r_t - u - \frac{Pu}{\rho\psi}\right] . \quad (29)$$

Examining the right-hand side of Eq. (20), we see that it also will equal zero if

$$r_t - u = \frac{Pu}{\rho\psi} , \quad (30)$$

$$z_t - w = \frac{Pw}{\rho\psi} . \quad (31)$$

Hence energy flows with speeds $(1+ P/\rho\psi)u$ and $(1+ P/\rho\psi)w$ in the radial and axial directions, respectively. It should be emphasized that all our dependent variables except r and z retain their same physical meaning as in the conventional Lagrangian formulation; specifically, u and w are still the mass particle speeds. However, through their dependence on the new independent variables ξ and η, they are correlated with the initial position of a given energy element. For impact problems in which one can assume that initially there is only kinetic energy of some projectile or moving medium, the initial position of a given energy element will be the same as the initial position of the corresponding mass element.

To complete the formulation of our problem, we must now ask what are the consequences of applying Eqs. (27), (30), and (31), to Eqs. (19), (21), and (22). Because

$$(\rho J)_t = \psi^{-1}[(\rho J \psi)_t - \rho J \psi_t] ,$$

and with Eq. (28) the left-hand side of Eq. (19) becomes

$$\psi^{-1}\left[-\rho_o\psi_o\,\alpha(\frac{\xi}{r})^\alpha\frac{r_t}{r} - \frac{\rho_o\psi_o}{\psi}(\frac{\xi}{r})^\alpha\psi_t + \alpha\rho_o\psi_o(\frac{\xi}{r})^\alpha\frac{u}{r}\right] ,$$

or

$$-\psi^{-1}\,\rho_o\psi_o(\frac{\xi}{r})^\alpha\left[\frac{\psi_t}{\psi} + \frac{\alpha}{r}(r_t - u)\right] .$$

Then substituting Eqs. (30) and (31) into the above and the right-hand side of Eq. (19), we have

$$\rho_o\psi_o(\frac{\xi}{r})^\alpha\left[(\psi^{-1})_t - \alpha\frac{Pu}{r\rho\psi^2}\right] = r_\xi(Pw\psi^{-1})_\eta - r_\eta(Pw\psi^{-1})_\xi + z_\eta(Pu\psi^{-1})_\xi - z_\xi(Pu\psi^{-1})_\eta \quad \text{mass} . \tag{32}$$

In Appendix C we repeat this procedure for Eqs. (21) and (22). The results are

$$\rho_o\psi_o(\frac{\xi}{r})^\alpha\left[(u\psi^{-1})_t - \alpha\frac{Pu^2}{r\rho\psi^2}\right] = r_\xi[Pwu\psi^{-1}]_\eta - r_\eta[Pwu\psi^{-1}]_\xi + z_\eta[Pu^2\psi^{-1}-P]_\xi - z_\xi[Pu^2\psi^{-1}-P]_\eta, \quad r \text{ momentum}, \tag{33}$$

$$\rho_o\psi_o(\frac{\xi}{r})^\alpha\left[(w\psi^{-1})_t - \alpha\frac{Puw}{r\rho\psi^2}\right] = r_\xi[Pw^2\psi^{-1}-P]_\eta - r_\eta[Pw^2\psi^{-1}-P]_\xi + z_\eta[Puw\psi^{-1}]_\xi - z_\xi[Puw\psi^{-1}]_\eta , \quad z \text{ momentum}. \tag{34}$$

Because Eq. (32) originally came from the mass conservation equation in Eulerian form, it should express mass conservation as one follows in time a particular energy element or cell, which is identified by its original position through (ξ,η). To show that this is indeed the case, we will rederive Eq. (32) integrally with the hope that its physical content will be more plausible. Equations (30) and (31) tell us that energy moves with radial and axial speeds, respectively, as

$$(1+\frac{P}{\rho\psi})u \text{ and } (1+\frac{P}{\rho\psi})w .$$

Hence, the velocity of a mass particle relative to the moving energy element is

$$\vec{v} - (1+\frac{P}{\rho\psi})\vec{v} = -\frac{P}{\rho\psi}\vec{v} .$$

Therefore, to express mass conservation, we write

$$\frac{d}{dt}\int_{V(t)}\rho dV = -\oiint_{S(t)} -\frac{P}{\psi}\vec{v}\,d\vec{S} , \tag{35}$$

where V is the moving volume element enclosed by the moving surface, S. By convention, $d\vec{S}$ points outward from the given energy element. The right-hand side is then the mass flux into a particular energy element; thus, the operator, d/dt, on the left-hand side implies following the given energy element in the fixed frame. In other words, Eqs. (30) and (31) imply that the rate of growth of the volume of a given energy element, i.e.,

$$J' = J(\frac{r}{\xi})^\alpha ,$$

is such that

$$\frac{J'_t}{J'} = \nabla\cdot(1+\frac{P}{\rho\psi})\vec{v} . \tag{36}$$

We think of the left-hand side of Eq. (36) as expressed in Lagrangian energy coordinates, (ξ,η,t), and of the right-hand side as a function of (r,z,t). Further, the operator, d/dt, in Eq. (35) is

$$\frac{d}{dt} = (\frac{\partial}{\partial t})_{r,z} + (1+\frac{P}{\rho\psi})\vec{v}\cdot\nabla , \tag{37}$$

and the transformation of this operator to Lagrangian energy coordinates is simply

$$\frac{d}{dt} \rightarrow (\frac{\partial}{\partial t})_{\xi,\eta} . \tag{38}$$

Equations (36) and (37) are derived in Appendix D. In the conventional Lagrangian coordinates, wherein mass is followed, the equations analogous to Eqs. (36) and (37) are

$$\frac{J'_t}{J'} = \nabla\cdot\vec{v} , \tag{39}$$

and

$$\frac{d}{dt} = (\frac{\partial}{\partial t})_{r,z} + \vec{v}\cdot\nabla . \tag{40}$$

To return to Eq. (35), we first rewrite is as

$$\frac{d}{dt}\int_{V(t)}\rho dV = \int_{V(t)}(\nabla\cdot\frac{P}{\psi}\vec{v})dV . \tag{41}$$

We then transform Eq. (41) to Lagrangian energy coordinates,

$$(\frac{\partial}{\partial t})_{\xi,\eta}\int_{V_o}\rho J'dV_o = \int_{V_o}\overline{(\nabla\cdot\frac{P}{\psi}\vec{v})}J'dV_o , \tag{42}$$

because $dV(r,z) = J'dV_o(\xi,\eta) = J'd\xi d\eta$. The bar over $\overline{\nabla\cdot P/\psi\ \vec{v}}$ reminds us that this operator is to be transformed to (ξ,η) coordinates. We now can take the operator, $(\partial/\partial t)_{\xi,\eta}$ on the left-hand side under the integral,

$$\int_{V_o} (\rho J')_t dV_o = \int_{V_o} \overline{(\nabla\cdot\frac{P}{\psi}\vec{v})} J'dV_o \ , \tag{43}$$

or

$$\int_{V_o} (\rho J')_t dV_o = \int_{V_o} \Big[\overline{(Pu\psi^{-1})}_r + \overline{(Pw\psi^{-1})}_z$$

$$+ \frac{\alpha}{r}\overline{(Pu\psi^{-1})}\Big]J'dV_o \ . \tag{44}$$

In the integrand on the left-hand side of Eq. (44), we substitute from Eq. (27),

$$\rho J' = \rho_o \psi_o \psi^{-1} \ , \tag{27a}$$

and carry out the transformations indicated on the right-hand side with Eqs. (11) and (12),

$$\int_{V_o} \rho_o\psi_o(\psi^{-1})_t dV_o = \int_{V_o} \Big\{ \Big[z_\eta(Pu\psi^{-1})_\xi - z_\xi(Pu\psi^{-1})_\eta$$

$$+ r_\xi(Pw\psi^{-1})_\eta - r_\eta(Pw\psi^{-1})_\xi\Big](\frac{r}{\xi})^\alpha + \frac{\alpha}{r}\ Pu\psi^{-1}\ J'\Big\}dV_o \ . \tag{45}$$

Because our volume element, dV_o, is arbitrary, the integrands in Eq. (45) may be equated, and we recognize that we have Eq. (32), after elimination of J' in the last term with Eq. (27a).

Finally, we can make one other check; we can go from Eq. (43) back to the fixed Eulerian frame with the help of Eqs. (36) through (38) to see whether we get Eq. (1). We first differentiate the integrand on the left-hand side of Eq. (43),

$$\int_{V_o} (\rho_t J' + \rho J'_t) dV_o = \int_{V_o} \overline{(\nabla\cdot\frac{P}{\psi}\vec{v})} J'dV_o \ , \tag{46}$$

and eliminate J'_t using Eq. (36),

$$\int_{V_o} \Big[\rho_t + \rho\nabla\cdot(1+\frac{P}{\rho\psi})\vec{v}\Big]J'dV_o = \int_{V_o} \overline{(\nabla\cdot\frac{P}{\psi}\vec{v})} J'dV_o \ . \tag{47}$$

We now transform to the fixed Eulerian frame, remembering that for ρ_t,

$$(\frac{\partial}{\partial t})_{\xi,\eta} \rightarrow \frac{d}{dt} \ ,$$

$$\int_{V(t)} \Big[\rho_t + (1+\frac{P}{\rho\psi})\vec{v}\cdot\nabla\rho + \rho\nabla\cdot(1+\frac{P}{\rho\psi})\vec{v} - \nabla\cdot\frac{P}{\psi}\vec{v}\Big]dV = 0. \tag{48}$$

Because the second and third terms can be combined as

$$\nabla\cdot\rho(1+\frac{P}{\rho\psi})\vec{v} = \nabla\cdot\rho v + \nabla\cdot\frac{P}{\psi}\vec{v} \ , \tag{49}$$

we see that we are left with Eq. (1).

The above arguments of plausibility are easily extended to the two momentum equations, Eqs. (33) and (34). For the r momentum, the equation of conservation is

$$\frac{d}{dt}\int_{V(t)} \rho u dV = -\oiint \rho u\Big[\vec{v} - (1+\frac{P}{\rho\psi})\vec{v}\Big]\cdot\vec{dS} - \oiint P(dS)^r, \tag{50}$$

where the first integral on the right-hand side is the r momentum flux into the energy element, and the second integral is the net force in the r-direction. Transforming to volume integrals on the right-hand side, we have

$$\frac{d}{dt}\int_{V(t)} \rho u dV = \int_{V(t)} \nabla\cdot\frac{Pu}{\psi}\vec{v}\ dV - \int_{V(t)} P_r\ dV \ . \tag{51}$$

Again, transforming to Lagrangian energy coordinates, we have

$$\int_{V_o} (\rho u J')_t dV_o = \int_{V_o} \Big[\overline{(Pu^2\psi^{-1})}_r + \overline{(Puw\psi^{-1})}_z$$

$$+ \overline{\frac{\alpha}{r}\ Pu^2\psi^{-1} - P_r}\Big]J'dV_o \ . \tag{52}$$

Eliminating J' through Eq. (27a) in the integrand on the left-hand side and in the third term on the right-hand side and carrying out the indicated transformation using Eqs. (11) and (12) on the right-hand side, we have

$$\int_{V_o} \rho_o\psi_o(u\psi^{-1})_t dV_o = \int_{V_o} \Big\{ \Big[z_\eta(Pu^2\psi^{-1})_\xi - z_\xi(Pu^2\psi^{-1})_\eta$$

$$+ r_\xi(Puw\psi^{-1})_\eta - r_\eta(Puw\psi^{-1})_\xi\Big](\frac{r}{\xi})^\alpha$$

$$+ \frac{\alpha}{r\rho}\ Pu^2\psi^{-2} - [z_\eta P_\xi - z_\xi P_\eta](\frac{r}{\xi})^\alpha\Big\}dV_o \ . \tag{53}$$

Because dV_o is arbitrary, Eq. (53) is Eq. (33).

To see if Eq. (52) yields Eq. (3) when transformed back to a fixed Eulerian frame, we first differentiate the integrand on the left-hand side,

$$\int_{V_o} [(\rho u)_t J' + \rho u J'_t] dV_o = \int_{V_o} \overline{\nabla \cdot \frac{Pu}{\psi} \vec{v}} J dV^o - \int_{V_o} \overline{P_r} \, J dV^o . \quad (54)$$

Substituting Eqs. (36) through (38) and transforming to (r,z,t) coordinates, we have

$$\int_{V(t)} \left[(\rho u)_t + (1+ \frac{P}{\rho\psi}) \vec{v} \cdot \nabla \rho u + \rho u \nabla \cdot (1+ \frac{P}{\rho\psi}) \vec{v} \right] dV$$

$$= \int_{V(t)} \nabla \cdot \frac{Pu}{\psi} \vec{v} \, dV - \int_{V(t)} P_r \, dV . \quad (55)$$

The second and third terms in the integrand on the left-hand side may be combined as

$$\nabla \cdot \rho u (1+ \frac{P}{\rho\psi}) \vec{v} = \nabla \cdot \rho u \vec{v} + \nabla \cdot \frac{Pu}{\psi} \vec{v} .$$

Hence, Eq. (55) becomes

$$\int_{V(t)} [(\rho u)_t + \nabla \cdot \rho u \vec{v}] dV = - \int_{V(t)} P_r \, dV , \quad (56)$$

which is Eq. (3) for arbitrary dV. The argument for Eq. (34), the z-momentum equation, follows identically to the above argument for the r momentum. Thus, Eqs. (32) through (34) are the fundamental differential equations to be solved in order to follow energy elements. The ψ^{-1} terms should be eliminated in finite differencing because there will be cells with no total energy in them. Some simplification can be achieved by subtracting from each momentum equation the corresponding speed times Eq.(32). Taking Eq. (33) and subtracting from it u times Eq. (32), we have

$$\rho_o \psi_o (\frac{\xi}{r})^{\alpha} u_t \psi^{-1} = r_\xi u_\eta P w \psi^{-1} - r_\eta u_\xi P w \psi^{-1}$$

$$+ z_\eta \left(u_\xi P u \psi^{-1} - P_\xi \right) - z_\xi \left(u_\eta P u \psi^{-1} - P_\eta \right),$$

or,

$$\rho_o \psi_o u_t = (\frac{r}{\xi})^{\alpha} \Big[Pw(r_\xi u_\eta - r_\eta u_\xi) + Pu(z_\eta u_\xi - z_\xi u_\eta)$$

$$- \psi(z_\eta P_\xi - z_\xi P_\eta) \Big] . \quad (57)$$

Similarly, subtracting w times Eq. (32) from Eq. (34), we have

$$\rho_o \psi_o w_t = (\frac{r}{\xi})^{\alpha} \Big[Pw(r_\xi w_\eta - r_\eta w_\xi) + Pu(z_\eta w_\xi - z_\xi w_\eta)$$

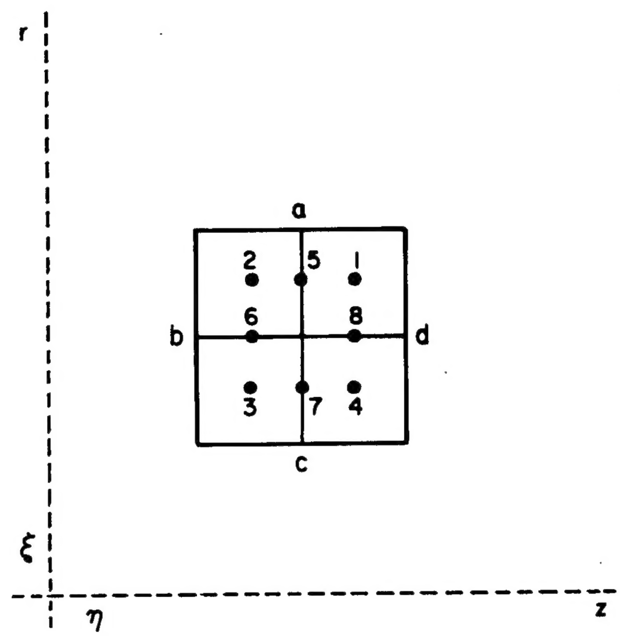
$$- \psi(r_\xi P_\eta - r_\eta P_\xi) \Big] . \quad (58)$$

Finally, we can eliminate ψ^{-1} in Eq. (32) by carrying out the differentiations and multiplying through by ψ^2. The resulting equation is,

$$\rho_o \psi_o \left(\psi_t + \alpha \frac{Pu}{r\rho} \right) = -(\frac{r}{\xi})^{\alpha} \Big\{ \psi \Big[r_\xi (Pw)_\eta - r_\eta (Pw)_\xi + z_\eta (Pu)_\xi - z_\xi (Pu)_\eta \Big]$$

$$- Pw \left(r_\xi \psi_\eta - r_\eta \psi_\xi \right) - Pu \left(z_\eta \psi_\xi - z_\xi \psi_\eta \right) \Big\} . \quad (59)$$

Equations (57) and (58) should be compared with Eqs. (25) and (26), their analogues in the conventional Lagrangian formulation; similarly, Eq. (59) should be compared with Eq. (24). Certainly the equations in Lagrangian energy coordinates are more complicated and their computations will require more machine time. When differenced, each acceleration equation will have three times as many factors as its conventional Lagrangian analogue. Because we are following energy cells, it would seem reasonable when differencing these equations to consider ψ as a cell-centered variable; hence, we ascribe a value for u and w at the energy cell center by averaging the values at the four corners. Further, because all the equations involve velocity gradients, it would make additional sense to have cell-centered velocities. If one examines some of the more popular methods of finite differencing the conventional formulas[2] and extends these methods to Eqs. (57) through (59), he will note that the mass accelerations, u_t, w_t, of a vertex point in the energy mesh depend through ψ on information within the four surrounding Lagrangian energy cells as well as at each corner of these cells. To illustrate, we give the explicit contribution to the r acceleration of cells whose midpoints are 1 and 4 in the following figure. That is, we difference Eq. (57) by assuming the acceleration in the r-direction of point 0 to be made up of four parts contributed by pairs of cells, 1 and 4, 2 and 3, 1 and 2, and 4 and 3. The contributions by cells 1 and 4 and 2 and 3 are averaged, as are the contributions by cells 1 and 2 and 4 and



3 (see Ref. 2, p. 18). The result is

$$u_t = \frac{(\bar{r}_8)^\alpha}{E_1+E_4}\Big\{(Pw)_8\Big[(r_1-r_4)(u_d-u_o)-(r_d-r_o)(u_1-u_4)\Big]$$

$$+ (Pu)_8\Big[(z_d-z_o)(u_1-u_4)-(z_1-z_4(u_d-u_o)\Big]$$

$$- \psi_8\Big[(z_d-z_o)(P_1-P_4)-(z_1-z_4)(P_d-P_o)\Big]\Big\}$$

$$+ \frac{(\bar{r}_6)^\alpha}{E_2+E_3} \;.\;.\;. + \frac{(\bar{r}_5)^\alpha}{E_1+E_2} \;.\;.\;. + \frac{(\bar{r}_7)^\alpha}{E_3+E_4} \;.\;.\;.\;,$$

where $\bar{r}_8 = \frac{1}{2}(r_d+r_o)$ or $\frac{1}{2}(r_1+r_4)$ and $E_i = (\rho_o\psi_o V_o)_i$ where V_o is the volume at $t = 0$ for cell i. In contrast, the same process for Eq. (25) yields

$$u_t = -\frac{(\bar{r}_8)^\alpha}{m_1+m_4}\Big[(P_1-P_4)(z_d-z_o)-(P_d-P_o)(z_1-z_4)\Big]$$

plus three similar terms, where $m_i = (\rho_o V_o)_i$. We have assumed that all dependent variables are known in some average way at all the lettered and numbered points in the figure. In practice, this could be achieved for points a, b, c, d by getting information from cells surrounding those depicted, or such terms could be ignored.

Interfaces between different materials will pose an additional complication here, because we will have to know them in order to choose the appropriate equation of state; i.e., in an energy cell that overlaps two or more mass cells, there will be contributions to the cell-centered pressure, density, and specific total energy from each of the mass constituents. Mass interfaces can be found, however, because u_t, w_t are the mass accelerations of a vertex point in the energy mesh. We would have to solve two additional equations,

$$r'_{tt} = u_t \quad \text{and} \quad z'_{tt} = w_t \;,$$

for instance, to follow mass points and then have to test r and z against these values to find the location of the energy points relative to the mass points. Certainly, it would be prudent to first try this method on a single-material problem.

Some remarks on the order of solving these equations are warranted, because this order demonstrates an additional difficulty not encountered in the conventional Lagrangian formulation. A proposed order of solution, assuming that all dependent variables except u and w are known at n, and that u and w are known at $n - \frac{1}{2}$, is:

1. Solve Eq. (59) for ψ^{n+1} ,
2. Solve Eq. (57) for $u^{n+\frac{1}{2}}$,
3. Solve Eq. (58) for $w^{n+\frac{1}{2}}$.

From these three equations, we can get

4. $e^{n+\frac{1}{2}} = \frac{1}{2}(\psi^{n+1}+\psi^n) - (u^2+w^2)^{n+\frac{1}{2}}$.
5. Solve $\rho^* = \dfrac{\rho_o\,\psi_o}{(J')^n\psi^{n+1}}$.
6. Assuming a Mie-Grüneisen equation of state, solve for,

$$P^* = f_1(\rho^*) + f_2(\rho^*)e^{n+\frac{1}{2}} \;.$$

7. $r^*-r^n = (\Delta t)^{n+\frac{1}{2}}\left(1+\dfrac{P^*}{\rho^*\psi^{n+\frac{1}{2}}}\right)u^{n+\frac{1}{2}}$,
8. $z^*-z^n = (\Delta t)^{n+\frac{1}{2}}\left(1+\dfrac{P^*}{\rho^*\psi^{n+\frac{1}{2}}}\right)w^{n+\frac{1}{2}}$,

where

$$(\Delta t)^{n+\frac{1}{2}} = t^{n+1} - t^n \;.$$

From r*, z* we now find a new value for J' to substitute into Eq. (5) and iterate on Eqs. (5) through (8) until consistency is attained between J', r*, z*. Call the final values r^{n+1}, z^{n+1},

$\rho^{n+\frac{1}{2}}$, $\rho^{n+\frac{1}{2}}$. Finally solve for

$$P^{n+1} = P^n + 2P^{n+\frac{1}{2}}, \quad \rho^{n+1} = \rho^n + 2\rho^{n+\frac{1}{2}} .$$

Thus, all quantities are advanced to $n + 1$ except u and w which are advanced to $n + \frac{1}{2}$.

Comment from any reader who has used this method in a two-dimensional code is invited.

ACKNOWLEDGMENT

I thank Mrs. Leah Peterson for typing this report.

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APPENDIX A. DERIVATION OF EQS. (11), (12), and (13)

Consider the transformation from (r,z,t) to (ξ,η,t') and the inverse,

$$\begin{aligned} r &= r(\xi,\eta,t') & \xi &= \xi(r,z,t) \\ z &= z(\xi,\eta,t') & \eta &= \eta(r,z,t) \\ t &= t(0,0,t') & t' &= t'(0,0,t) , \end{aligned}$$

such that $t = t'$. We retain the prime on t associated with the ξ, η, t' set of independent variables for convenience. The operators of partial differentiation are related by

$$\begin{aligned} \frac{\partial}{\partial r} &= \eta_r \frac{\partial}{\partial \eta} + \xi_r \frac{\partial}{\partial \xi} , \\ \frac{\partial}{\partial z} &= \eta_z \frac{\partial}{\partial \eta} + \xi_z \frac{\partial}{\partial \xi} , \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} + \eta_t \frac{\partial}{\partial \eta} + \xi_t \frac{\partial}{\partial \xi} . \end{aligned} \tag{A-1}$$

We think of the coefficients in the above equations, η_r, ξ_r, η_z, ξ_z, η_t, and ξ_t, as functions of r, z, t, so we must find the transform of these functions in order to get the operators on the right-hand side as functions of our new independent variables ξ, η, t. To accomplish this, we consider how the differentials transform.

$$\begin{pmatrix} dr \\ dz \\ dt \end{pmatrix} = \begin{pmatrix} r_\xi & r_\eta & r_{t'} \\ z_\xi & z_\eta & z_{t'} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \\ dt' \end{pmatrix} \tag{A-2}$$

and the inverse,

$$\begin{pmatrix} d\xi \\ d\eta \\ dt' \end{pmatrix} = \begin{pmatrix} \xi_r & \xi_z & \xi_t \\ \eta_r & \eta_z & \eta_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dr \\ dz \\ dt \end{pmatrix} . \tag{A-3}$$

Now we can invert Eq. (A-2) symbolically.

$$\begin{pmatrix} d\xi \\ d\eta \\ dt' \end{pmatrix} = |a_{ij}|^{-1} (\text{cofactor of } a_{ij} \text{ transposed}) \begin{pmatrix} dr \\ dz \\ dt \end{pmatrix} , \tag{A-4}$$

where a_{ij} is the matrix in Eq. (A-2) and $|a_{ij}|$ is the determinant of a_{ij}. Explicitly, Eq. (A-4) is

$$\begin{pmatrix} d\xi \\ d\eta \\ dt' \end{pmatrix} = |a_{ij}|^{-1} \begin{pmatrix} z_\eta & -r_\eta & r_\eta z_t - r_t z_\eta \\ -z_\xi & r_\xi & -(r_\xi z_t - r_t z_\xi) \\ 0 & 0 & r_\xi z_\eta - r_\eta z_\xi \end{pmatrix} \begin{pmatrix} dr \\ dz \\ dt \end{pmatrix} . \tag{A-5}$$

Equations (A-3) and (A-5) demand that the corresponding elements in their respective matrices be equal. In addition, we see that

$$|a_{ij}| = J$$

in our previous notation. Hence,

$$\begin{aligned} \xi_r &= J^{-1} z_\eta, & \xi_z &= -J^{-1} r_\eta, & \xi_t &= J^{-1}(r_\eta z_t - r_t z_\eta) \\ \eta_r &= -J^{-1} z_\xi, & \eta_z &= J^{-1} r_\xi, & \eta_t &= -J^{-1}(r_\xi z_t - r_t z_\xi). \end{aligned} \tag{A-6}$$

Substituting Eqs. (A-6) into Eqs. (A-1) we have Eqs. (11), (12), and (13) in the body of this report.

APPENDIX B. DERIVATION OF EQS. (20, (21) and (22)

Applying Eqs. (11), (12), and (13) to Eq. (6) yields

$$\begin{aligned} &(\rho\psi)_t J + (r_t z_\xi - r_\xi z_t)(\rho\psi)_\eta + (r_\eta z_t - r_t z_\eta)(\rho\psi)_\xi + \alpha J(\rho\psi + P)\frac{u}{r} \\ &+ z_\eta[(\rho\psi+P)u]_\xi - z_\xi[(\rho\psi+P)u]_\eta + r_\xi[(\rho\psi+P)w]_\eta - r_\eta[(\rho\psi+P)w]_\xi = 0. \end{aligned} \tag{B-1}$$

Into Eq. (B-1), substitute

$$(\rho\psi)_t J = (\rho\psi J)_t - \rho\psi(r_\xi z_{\eta t} + r_{\xi t} z_\eta - r_\eta z_{\xi t} - r_{\eta t} z_\xi) , \tag{B-2}$$

and then collect terms in r_ξ, r_η, z_η, z_ξ. The result is

$$(\rho\psi J)_t+\alpha J(\rho\psi+P)\frac{u}{r} = r_\xi\left\{\rho\psi z_{\eta t}+(\rho\psi)_\eta z_t-[(\rho\psi+P)w]_\eta\right\}$$
$$- r_\eta\left\{\rho\psi z_{\xi t}+(\rho\psi)_\xi z_t-[(\rho\psi+P)w]_\xi\right\}$$
$$+ z_\eta\left\{\rho\psi r_{\xi t}+(\rho\psi)_\xi r_t-[(\rho\psi+P)u]_\xi\right\}$$
$$- z_\xi\left\{\rho\psi r_{\eta t}+(\rho\psi)_\eta r_t-[(\rho\psi+P)u]_\eta\right\} , \quad \text{(B-3)}$$

which is Eq. (20).

Repeating this procedure on Eq. (7),

$$(\rho u)_t J+(r_t z_\xi-r_\xi z_t)(\rho u)_\eta+(r_\eta z_t-r_t z_\eta)(\rho u)_\xi+\alpha J\ \frac{\rho u^2}{r}$$
$$+z_\eta(\rho u^2+P)_\xi-z_\xi(\rho u^2+P)_\eta+r_\xi(\rho uw)_\eta-r_\eta(\rho uw)_\xi = 0; \quad \text{(B-4)}$$

$$(\rho u)_t J = (\rho uJ)_t - \rho u\ J_t\ ; \quad \text{(B-5)}$$

$$(\rho uJ)_t+\alpha J\ \frac{\rho u^2}{r} = r_\xi\left[\rho uz_{\eta t}+(\rho u)_\eta z_t-(\rho uw)_\eta\right]$$
$$-r_\eta\left[\rho uz_{\xi t}+(\rho u)_\xi z_t-(\rho uw)_\xi\right]+z_\eta\left[\rho ur_{\xi t}+(\rho u)_\xi r_t -(\rho u^2+P)_\xi\right]$$
$$-z_\xi\left[\rho ur_{\eta t}+(\rho u)_\eta r_t-(\rho u^2+P)_\eta\right] , \quad \text{(B-6)}$$

which is Eq. (21).

Finally, repeating this procedure on Eq. (8),

$$(\rho w)_t J+(r_t z_\xi-r_\xi z_t)(\rho w)_\eta+(r_\eta z_t-r_t z_\eta)(\rho w)_\xi+\alpha J\ \frac{\rho wu}{r}$$
$$+z_\eta(\rho wu)_\xi-z_\xi(\rho wu)_\eta+r_\xi(\rho w^2+P)_\eta-r_\eta(\rho w^2+P)_\xi = 0 \quad ; \quad \text{(B-7)}$$

$$(\rho w)_t J = (\rho wJ)_t - \rho wJ_t\ ; \quad \text{(B-8)}$$

$$(\rho wJ)_t+\alpha J\ \frac{\rho wu}{r} = r_\xi\left[\rho wz_{\eta t}+(\rho w)_\eta z_t-(\rho w^2+P)_\eta\right]$$
$$-r_\eta\left[\rho wz_{\xi t}+(\rho w)_\xi z_t-(\rho w^2+P)_\xi\right]+z_\eta\left[\rho wr_{\xi t}+(\rho w)_\xi r_t - (\rho wu)_\xi\right]$$
$$- z_\xi\left[\rho wr_{\eta t}+(\rho w)_\eta r_t-(\rho wu)_\eta\right] , \quad \text{(B-9)}$$

which is Eq. (22).

APPENDIX C. DERIVATION OF EQS. (33) AND (34)

With Eq. (27), the left-hand side of Eq. (21) becomes

$$\frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha u_t + \left[\frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha\right]_t u + \alpha\ \frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha \frac{u^2}{r} , \quad \text{(C-1)}$$

or

$$\frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha u_t - \rho_o\psi_o\left[\frac{\alpha}{\psi}\left(\frac{\xi}{r}\right)^\alpha \frac{r_t}{r} + \frac{\psi_t}{\psi^2}\left(\frac{\xi}{r}\right)^\alpha\right]u+\alpha\ \frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha \frac{u^2}{r} , \quad \text{(C-2)}$$

or

$$\frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha\left[u_t - \frac{\psi_t}{\psi}\ u-\alpha(r_t-u)\frac{u}{r}\right] . \quad \text{(C-3)}$$

Then, with Eqs. (30) and (31), Eq. (21) is

$$\rho_o\psi_o\left(\frac{\xi}{r}\right)^\alpha\ \psi^{-1}\left[u_t-\psi^{-1}\psi_t u - \alpha\psi^{-1}\ \frac{Pu^2}{r\rho}\right]$$
$$= r_\xi(Pwu\psi^{-1})_\eta - r_\eta(Pwu\psi^{-1})_\xi$$
$$+ z_\eta(Pu^2\psi^{-1}-P)_\xi - z_\xi(Pu^2\psi^{-1}-P)_\eta , \quad \text{(C-4)}$$

which is Eq. (33).

Repeating the above procedure on Eq. (22), the left-hand side is

$$\frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha\ w_t - \rho_o\psi_o\left[\frac{\alpha}{\psi}\left(\frac{\xi}{r}\right)^\alpha \frac{r_t}{r} + \frac{\psi_t}{\psi^2}\left(\frac{\xi}{r}\right)^\alpha\right]w$$
$$+ \alpha\ \frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha \frac{wu}{r} , \quad \text{(C-5)}$$

or

$$\frac{\rho_o\psi_o}{\psi}\left(\frac{\xi}{r}\right)^\alpha\left[w_t - \frac{\psi_t}{\psi}\ w - \alpha(r_t-u)\frac{w}{r}\right] . \quad \text{(C-6)}$$

Then, with Eqs. (30) and (31), Eq. (22) is

$$\rho_o\psi_o\left(\frac{\xi}{r}\right)^\alpha\ \psi^{-1}\left(w_t-\psi^{-1}\psi_t w-\alpha\psi^{-1}\ \frac{Puw}{r\rho}\right) = r_\xi(Pw^2\psi^{-1}-P)_\eta$$
$$-r_\eta(Pw^2\psi^{-1}-P)_\xi + z_\eta(Puw\psi^{-1})_\xi - z_\xi(Puw\psi^{-1})_\eta , \quad \text{(C-7)}$$

which is Eq. (34).

APPENDIX D. DERIVATION OF EQS. (36) AND (37) FROM EQS. (30) AND (31)

By definition,

$$J' = \frac{\partial(r,z)}{\partial(\xi,\eta)} \left(\frac{r}{\xi}\right)^{\alpha} = J\left(\frac{r}{\xi}\right)^{\alpha} , \qquad \text{(D-1)}$$

$$J' = \left(\frac{r}{\xi}\right)^{\alpha} \begin{vmatrix} r_\xi & r_\eta \\ z_\xi & z_\eta \end{vmatrix} . \qquad \text{(D-2)}$$

Differentiating Eq. (D-2) with respect to t,

$$J'_t = \left(\frac{r}{\xi}\right)^{\alpha} \begin{vmatrix} r_{\xi t} & r_{\eta t} \\ z_\xi & z_\eta \end{vmatrix} + \left(\frac{r}{\xi}\right)^{\alpha} \begin{vmatrix} r_\xi & r_\eta \\ z_{\xi t} & z_{\eta t} \end{vmatrix}$$

$$+ \alpha\left(\frac{r}{\xi}\right)^{\alpha-1} \frac{r_t}{\xi} \begin{vmatrix} r_\xi & r_\eta \\ z_\xi & z_\eta \end{vmatrix} . \qquad \text{(D-3)}$$

Into Eq. (D-3) we now substitute Eqs. (30) and (31),

$$r_t = \left(1+ \frac{P}{\rho\psi}\right)u, \qquad z_t = \left(1+ \frac{P}{\rho\psi}\right)w \quad ,$$

and for convenience let $1 + \frac{P}{\rho\psi} = \phi$. Then,

$$J'_t = \left(\frac{r}{\xi}\right)^{\alpha} \begin{vmatrix} (\phi u)_\xi & (\phi u)_\eta \\ z_\xi & z_\eta \end{vmatrix} + \left(\frac{r}{\xi}\right)^{\alpha} \begin{vmatrix} r_\xi & r_\eta \\ (\phi w)_\xi & (\phi w)_\eta \end{vmatrix}$$

$$+ \alpha\left(\frac{r}{\xi}\right)^{\alpha-1} \frac{\phi u}{\xi} J \quad . \qquad \text{(D-4)}$$

Now consider ϕu and ϕw as functions of r and z; then, by the chain rule,

$$(\phi u)_\xi = r_\xi(\phi u)_r + z_\xi(\phi u)_z \quad ; \qquad \text{(D-5)}$$

$$(\phi u)_\eta = r_\eta(\phi u)_r + z_\eta(\phi u)_z \quad ; \qquad \text{(D-6)}$$

$$(\phi w)_\xi = r_\xi(\phi w)_r + z_\xi(\phi w)_z \quad ; \qquad \text{(D-7)}$$

$$(\phi w)_\eta = r_\eta(\phi w)_r + z_\eta(\phi w)_z \quad . \qquad \text{(D-8)}$$

Substituting the above relations into Eq. (D-4), and in the last term substituting

$$J = J'/\left(\frac{r}{\xi}\right)^{\alpha} , \qquad \text{(D-9)}$$

we have

$$J'_t = \left(\frac{r}{\xi}\right)^{\alpha} \left\{ (\phi u)_r \begin{vmatrix} r_\xi & r_\eta \\ z_\xi & z_\eta \end{vmatrix} + (\phi u)_z \begin{vmatrix} z_\xi & z_\eta \\ z_\xi & z_\eta \end{vmatrix} \right.$$

$$\left. + (\phi w)_r \begin{vmatrix} r_\xi & r_\eta \\ r_\xi & r_\eta \end{vmatrix} + (\phi w)_z \begin{vmatrix} r_\xi & r_\eta \\ z_\xi & z_\eta \end{vmatrix} \right\} + \alpha \frac{\phi u}{r} J' ,$$

$$\text{(D-10)}$$

or

$$J'_t = [(\phi u)_r + \frac{\alpha}{r} \phi u + (\phi w)_z]J' , \qquad \text{(D-11)}$$

which is Eq. (36).

For Eq. (37), we first rewrite Eq. (13) in the body of report in the form, using Eqs. (11) and (12),

$$\left(\frac{\partial}{\partial t}\right)_{r,z} = \left(\frac{\partial}{\partial t}\right)_{\xi,\eta} - \left(\frac{\partial r}{\partial t}\right)_{\xi,\eta} \left(\frac{\partial}{\partial r}\right)_{z,t} - \left(\frac{\partial z}{\partial t}\right)_{\xi,\eta} \left(\frac{\partial}{\partial z}\right)_{r,t}$$

$$\text{(D-12)}$$

or

$$\left(\frac{\partial}{\partial t}\right)_{\xi,\eta} = \left(\frac{\partial}{\partial t}\right)_{r,z} + \left(\frac{\partial r}{\partial t}\right)_{\xi,\eta} \left(\frac{\partial}{\partial r}\right)_{z,t} + \left(\frac{\partial z}{\partial t}\right)_{\xi,\eta} \left(\frac{\partial}{\partial z}\right)_{r,t} .$$

$$\text{(D-13)}$$

Then, substituting Eqs. (30) and (31) into the above, we have

$$\left(\frac{\partial}{\partial t}\right)_{\xi,\eta} = \frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_{r,z} + \left(1+ \frac{P}{\rho\psi}\right)\vec{v}\cdot\nabla , \qquad \text{(D-14)}$$

which is Eq. (37).